Actions

Of the various formalisms for describing mechanics, the Lagrangian and associated action principle is the most useful for our purposes since it can be generalized in several ways:

$\underbrace{NR \to \text{Relativistic}}_{\int dt \,\Rightarrow\, \int d^4x}$, $\underbrace{PP \to \text{Fields}}_{x(t) \,\to\, \phi(x^\mu)}$, $\underbrace{CM \to QM}_{\delta S = 0 \,\to\, \int D\phi \, e^{\frac{i}{\hbar}S}}$

and because it allows us to work with manifest symmetries.

Preliminary:

The action is a <u>functional</u> which essentially means a function of functions.

For example, whereas a function takes an argument and returns a number: $f(x) = x^2 \Rightarrow f(2) = 4$
a functional does the same when we insert a function: $S[f(x)] = \int_0^1 f(x)dx$

$$\Downarrow$$

$$S[x^2] = \int_0^1 x^2 dx = \frac{1}{3}$$

What makes this nice is that in the same way we can <u>extremize</u> a function w/ $\frac{df}{dx} = 0$ to find the value of x which achieves the max or min of $f(x)$, we can also extremize a functional w/ $\frac{\delta S}{\delta f} = 0$ to find the function $f(x)$ which achieves the max or min of $S[f(x)]$.

You should be familiar w/ $S = \int L\,dt$ $\quad L(q, \dot{q}) = T - V$ and $\delta S = 0 \Rightarrow \underbrace{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0}_{\text{Euler-Lagrange e.o.m.}}$

We need a relativistic version of this suitable for quantum use.

Action, Lagrangian & E.O.M. for Relativistic Fields

Replace: $q(t) \to \phi(x^\mu)$, $\dot{q}(t) \to \frac{\partial \phi}{\partial x^\mu}$, $\int (\ )dt \to \int (\ \ )d^4x$

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x$$

Eventually we will incorporate this in a path integral, but it also turns out that the classical e.o.m. will be useful.

Consider $\phi(x^\mu)$ and a region M of spacetime w/ boundary conditions specified $\phi(x^\mu|_{\partial M})$

Now consider a deformed field configuration $\phi'(x^\mu) = \phi(x^\mu) + \delta\phi(x^\mu)$ that satisfies the same b.c.s., i.e. $\phi'(x^\mu|_{\partial M}) = \phi(x^\mu|_{\partial M}) \Rightarrow \delta\phi(x^\mu|_{\partial M}) = 0$

The classical field configuration satisfies $\delta S = \int \delta\mathcal{L}(\phi, \partial_\mu\phi)\, d^4x = 0$

$$\delta S = \int \Big[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi)} \Big] = 0$$

I.B.P. using $\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\partial_\mu\phi$

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\delta\phi\, d^4x + \underbrace{\int \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)d^4x}_{= \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\big|_{\partial M} = 0 \text{ by}} = 0$$

For this to be true for arbitrary $\delta\phi$ we need: $\frac{\partial \mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$ EL e.o.m.

compare to: $\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$

If you are familiar with Lagrangian mechanics then you are probably used to constructing the Lagrangian with the kinetic and potential energy of the degrees of freedom according to

$$L = \underbrace{T}_{\text{kinetic energy}} - \underbrace{V}_{\text{Potential energy}}$$

The idea of potential energy is useful, but for our purposes it is better to directly associate potential energies with the interactions between things (fields in our case).

So generically we expect the Lagrangian to split into: $\mathcal{L} = \mathcal{L}_{kinetic} + \mathcal{L}_{interaction}$

This sign is not that important since we haven't yet specified how to construct $\mathcal{L}_{interaction}$

Now if we have no interactions, then we have what is called a free theory in which case $\mathcal{L}_{free} = \mathcal{L}_{kinetic}$ so sometimes we call the kinetic term the "free Lagrangian".

The program we will follow is to first consider free Lagrangians and then introduce interactions through the principle of local gauge invariance.

Free Lagrangians

In classical point particle physics you would say $L_{free} = T = \frac{p^2}{2m} = \frac{1}{2}mv^2$.

However this is based on a massive scalar particle and built from the 3-momentum $\vec{p}$ (or velocity $\vec{v}$).

All of this changes for (possibly massless) relativistic fields.

Fortunately we only have 3 cases to consider: $\underbrace{\text{spin-}0}_{\text{Higgs}}$, $\underbrace{\text{spin-}\frac{1}{2}}_{\text{matter}}$, $\underbrace{\text{spin-}1}_{\text{force particles}}$

We won't derive these free Lagrangians. Doing so can be done in various ways at different levels of sophistication and to be honest several of them were actually guessed in their original discovery. We will just present them one at a time, then derive the Euler-Lagrange equation of motion

Spin-0 (scalars) ϕ

$$\mathcal{L}_{free} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2$$

Notice that the mass term is separate which allows us to consistently handle $m=0$ cases.

$$= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2$$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

$$\left(\frac{mc}{\hbar}\right)^2\phi - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial}{\partial x^\nu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) = 0$$

$$\left(\frac{mc}{\hbar}\right)^2\phi - \frac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{1}{2}\eta^{\mu\nu}\partial_\nu\partial_\mu\phi = 0$$

$$\left(\frac{mc}{\hbar}\right)^2\phi - \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$$

$$\partial_\mu\partial^\mu\phi - \left(\frac{mc}{\hbar}\right)^2\phi = 0$$

The Klein-Gordon Equation

Spin-1 (vectors) A^μ

$$\mathcal{L}_{free} = \frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\mu A_\mu \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \text{ and } F^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\rho}F_{\lambda\rho}$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{1}{4\pi}\left(\frac{mc}{\hbar}\right)^2 A^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{1}{4\pi} F^{\mu\nu}$$ (You get to fill in the details in your homework)

Then: $\partial_\mu F^{\mu\nu} - \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0$ The Proca Equation

Taking $m^2 = 0$ we have $\partial_\mu F^{\mu\nu} = 0 \Rightarrow$

usually $\vec{J}$

$$\left.\begin{array}{l} \vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t} = 0 \\ \vec{\nabla}\cdot\vec{E} = 0 \end{array}\right\} \text{½ of Maxwell's equations}$$

usually ρ

Note: $F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$

$$\left.\begin{array}{l} \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \end{array}\right\} \text{Non-dynamical, come from electromagnetic "geometry"}$$

Spin-$\frac{1}{2}$ (spinors) ψ & $\bar{\psi}$

$\mathcal{L}_{free} = (\hbar c)\bar{\psi}\gamma^\mu\partial_\mu\psi + mc^2\bar{\psi}\psi$ We treat ψ & $\bar{\psi}$ as independent degrees of freedom for reasons to be discussed.

Varying w.r.t. $\bar{\psi}$:

$$\underbrace{\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right)}=0$$

$$\hbar \underbrace{\gamma^\mu\partial_\mu}_{\equiv\not{\partial}}\psi + mc^2\psi = 0 \Rightarrow \not{\partial}\psi + \frac{mc}{\hbar}\psi = 0$$ The Dirac Equation

Interesting...

Varying w.r.t. ψ:

$$\underbrace{\frac{\partial\mathcal{L}}{\partial\psi}} - \underbrace{\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right)}=0$$

$$mc^2\bar{\psi} - \partial_\mu(\hbar c\bar{\psi}\gamma^\mu) = 0 = \not{\partial}\bar{\psi} - \frac{mc}{\hbar}\bar{\psi} = 0$$ (The adjoint of the Dirac Equation)

Consider the 3 equations of motion we have discussed so far:

Spin 0: $\partial_\mu \partial^\mu \phi - \left(\frac{mc}{\hbar}\right)^2 \phi = 0$ K-G equation

Spin 1/2: $\not{\partial}\psi + \frac{mc}{\hbar}\psi = 0 \Rightarrow$ You showed in your HW that $\Rightarrow \overbrace{\partial_\mu \partial^\mu \psi - \left(\frac{mc}{\hbar}\right)^2 \psi = 0}^{\text{K-G equation}}$

holds for each component of ψ and A^ν separately!

Spin 1: $\partial_\mu F^{\mu\nu} - \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0 \Rightarrow \underbrace{\partial_\nu \partial_\mu F^{\mu\nu}}_{=0} - \left(\frac{mc}{\hbar}\right)^2 \partial_\nu A^\nu = 0 \Rightarrow \partial_\nu A^\nu = 0$

since sym · antisym

$$\Rightarrow \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) - \left(\frac{mc}{\hbar}\right)^2 A^\nu = \underbrace{\partial_\mu \partial^\mu A^\nu - \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0}_{\text{K-G equation}}$$

Why does everything also satisfy the Klein-Gordon equation?

Consider: $\frac{E^2}{c^2} - p^2 = m^2c^2 \Rightarrow p_\mu p^\mu + m^2c^2 = 0$ let $p_\mu = i\hbar\partial_\mu$ and act on ϕ

$$-\hbar^2 \partial_\mu \partial^\mu \phi + m^2c^2\phi = 0 \Rightarrow \underbrace{\partial_\mu \partial^\mu \phi - \left(\frac{mc}{\hbar}\right)^2 \phi = 0}$$

K-G equation is really just reflecting the mass-shell condition which all real degrees of freedom must satisfy!

Note: Starting w/ $\frac{p^2}{2m} + V = E$

using $p \to -i\hbar\vec{\nabla}$, $E \to i\hbar\frac{\partial}{\partial t}$

then $\underbrace{-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}}$

Non-relativistic Time-dependent Schrodinger Equation